2016

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Abstract

This paper presents a quadratic-program-based framework for group decision making with incomplete intuitionistic preference relations (IPRs). The framework starts with introducing a notion of additive consistency for incomplete IPRs, followed by a two-stage quadratic program model for estimating missing values in an incomplete IPR. The first stage aims to minimize inconsistency of the completed IPR and control hesitation margins of the estimated judgments within an acceptable threshold. The second stage is to find the most suitable estimates without changing the inconsistency level. Subsequently, a parameterized formula is proposed to transform normalized interval fuzzy weights into additively consistent IPRs. Two quadratic programs are developed to generate interval fuzzy weights from a complete IPR. The first model obtains interval fuzzy weight vectors by minimizing the squared deviation between the two sides of the transformation formula. By optimizing the parameter value, the second model finds the best weight vector based on the optimal solutions of the first model. A procedure is then developed to solve group decision problems with incomplete IPRs. A numerical example and a group selection problem for enterprise resource planning software products are provided to demonstrate the proposed models.

Keywords: Intuitionistic preference relation, Additive transitivity, Quadratic program, Completion, Group decision making

1. Introduction

In multi-criteria decision making (MCDM), decision-makers (DMs) often employ pairwise comparison to elicit their preference over alternatives. These preference judgments are structured as multiplicative preference relations in the classic analytic hierarchy process (AHP) (Saaty, 1980). To express DMs' pairwise judgments with vagueness, Orlovski (1978) introduced fuzzy preference relations, which is also referred to as reciprocal preference relations (De Baets & De Meyer, 2005; Chiclana et al., 2009). Crisp-ratio and unit-interval bipolar scales are two most commonly used approaches in representing a DM's pairwise comparison results. The classical AHP adopts a crisp-ratio approach where the numerical value 1 plays a neutral role in representing the DM's indifference between two alternatives. On the other hand, a unit-interval bipolar scale uses the numerical value 0.5 to express its neutral value. This scale has been widely applied to decision models with [0, 1]-valued reciprocal preference relations and [0, 1]-valued interval reciprocal preference relations. It is noted that there exists an isomorphism between a unit-interval bipolar scale with the numerical value 0.5 and a crisp-ratio bipolar scale with the neutral value 1.

A variety of methods have been put forward to generate priority weights from fuzzy preference relations and estimate missing values for incomplete fuzzy preference relations. For instance, Xu (2004) introduced additive consistency and multiplicative consistency for incomplete fuzzy preference relations and developed two goal programs for obtaining priority weights from incomplete fuzzy preference relations. Herrera-Viedma et al. (2007) introduced an additive consistency index to define the inconsistency level of a fuzzy preference relation, and put forward an iterative procedure to estimate unknown values for incomplete fuzzy preference relations. Liu et al. (2012) developed a least square model to determine missing values for incomplete fuzzy preference relations based on additive transitivity.

An element in a fuzzy preference relation represents a DM's judgment with a membership degree. Sometimes, DMs may have hesitancy or uncertainty for their membership judgments. In this situation, Atanassov (1986)'s intuitionistic fuzzy sets (A-IFSs) appears to be a convenient representation framework. A-IFSs employ both membership and nonmembership functions to characterize DMs' vague judgments, and have been widely applied to areas such as decision making (Qi et al., 2015; İntepe et al., 2013; Xu & Liao, 2014), clustering analysis (Chaira, 2011) and machine learning (Szmidt et al., 2014). Since Xu (2007) introduced the notion of intuitionistic preference relations (IPRs), decision modeling with IPRs has attracted attention from many researchers in recent years (Jiang et al., 2015; Xu & Liao, 2014; Yue & Jia, 2015).

Based on various transitivity conditions, some approaches have been devised to estimate missing values in incomplete IPRs and obtain priority weights from complete IPRs. For instance, Xu et al. (2011) introduced a multiplicative transitivity equation to define consistency of IPRs and proposed two algorithms to determine missing elements for incomplete IPRs. Gong et al. (2009) established goal programming models for deriving interval priority weights from IPRs. Xu (2012) put forward an approach to determine interval weights of IPRs based on an error analysis idea. Recently, Xu and Liao (2014) extended crisp and fuzzy AHPs to the intuitionistic AHP and developed a normalizing rank summation method to obtain priority weights from IPRs.

Wu and Chiclana (2014) proposed a different multiplicative consistency definition for IPRs and develop a consistency based procedure to estimate missing values. Wang (2015) revealed that the multiplicative consistency given by Xu et al. (2011) has an undesirable property: the same IPR's consistency status may change when the alternatives are re-labeled. A geometric consistency definition is proposed for IPRs to address this issue. A logarithmic-least-square optimization model was also developed to elicit interval fuzzy weights from IPRs.

Chiclana et al. (2009) converted Tanino (1984)'s multiplicative transitivity constraint to an equivalent Cross Ratio uninorm based functional equation for fuzzy preference relations, and indicated that the uninorm-based function is more appropriate to tackle the boundary problem for consistency of reciprocal preference relations. However, as an alternative notion, additive consistency remains a viable choice to characterize whether pairwise comparison judgments are consistent and was adopted in recent research (Cabrerizo et al., 2010; Meng and Chen, 2015; Zhang et al., 2014). As Xu et al. (2014) pointed out, the uninorm-based function does not perform well and may yield counterintuitive consistent judgment when a furnished preference value approaches 0 or 1. On the other hand, additive transitivity behaves well with intuitionistic judgments close to $(1,0)$ and $(0,1)$. The authors contemplate that additive and multiplicative consistency might reflect different human cognitive characteristics when they provide their pairwise judgments: For linear-thinking-inclined DMs, additive consistency is more appropriate, but for nonlinear thinking DMs, multiplicative consistency appears to be a better choice. The research herein adopts the notion of additive consistency.

Under the framework of additive consistency, Xu (2009) introduced a feasible region method to define additively consistent IPRs and established a linear program to obtain a priority weight vector from an IPR. Gong et al. (2011) presented a goal program and a least square model for deriving interval fuzzy weights from IPRs. Wang (2013) introduced a new transitivity condition to define additively consistent IPRs and developed two goal programs for deriving intuitionistic fuzzy priority weight vectors from IPRs. In Gong et al. (2011) and Wang (2013), the coefficient of the transformation formulae between additively consistent IPRs and priority weights is assumed to be 0.5, same as that of Tanino (1984)'s additive transformation formula. It has been found that this transformation relation is not always valid (Fedrizzi and Brunelli, 2009; Liu et al., 2012; Xu et al., 2009; 2010; 2014; Hu et al., 2014). This motivates us to introduce a parameterized transformation formula between additively consistent IPRs and priority weights and develop a corresponding priority weight derivation method.

This research first extends the additive consistency for IPRs to the case of incomplete IPRs. A two-stage quadratic program framework is then put forward to estimate missing values in incomplete IPRs. The first stage minimizes the inconsistency level of the completed IPR with an appropriate control of hesitation margins of the estimated judgments. The second stage finds the most suitable estimated values among the results obtained from the first stage without changing the inconsistency level. By analyzing the inherent relationship between an additively consistent IPR (Wang, 2013) and a normalized interval fuzzy weight vector and introducing a parameterized transformation formula, two quadratic programs are developed to obtain a normalized interval fuzzy weight vector. The first model minimizes the squared deviations between the original intuitionistic judgments and the parameterized interval-weight-based preference values. The second model identifies the most appropriate interval fuzzy weight vector among the optimal solutions in the first model by optimizing the parameter value. Finally, by applying the aforesaid models, a procedure is developed for solving group decision making (GDM) problems with incomplete IPRs.

The remainder of the paper is organized as follows. Section 2 reviews basic concepts of additively consistent fuzzy preference relations and IPRs. Section 3 introduces the notion of additive consistency for incomplete IPRs, and devises a two-stage approach to estimate missing values in incomplete IPRs. Two quadratic programs are proposed for generating interval fuzzy weights from complete IPRs in Section 4. Section 5 puts forward a practical procedure to solve GDM problems with incomplete IPRs, followed by a numerical illustration. Conclusions are drawn in Section 6.

2. Preliminaries

This section presents basic concepts of additively consistent fuzzy preference relations and IPRs.

Let $X=\{x_1,x_2,...,x_n\}$ be a collection of n alternatives. A fuzzy preference relation (Orlovski, 1978) on X is defined by a pairwise judgment matrix $R=(r_{ij})_{n\times n}$, where r_{ij} indicates a DM's fuzzy preference of alternative x_i over x_j such that

$$r_{ij}\in[0,1], r_{ij}+r_{ji}=1, r_{ii}=0.5, \quad \forall i, j=1,2,...,n \tag{2.1}$$

Definition 2.1 (Tanino, 1984) A fuzzy preference relation $R=(r_{ij})_{n\times n}$ is additively consistent if R satisfies additive transitivity:

$$r_{ij}=r_{ik}-r_{kj}+0.5,\quad \forall i,j,k=1,2,\ldots,n. \tag{2.2}$$

Due to additive reciprocity $r_{ij}+r_{ji}=1$, (2.2) is equivalent to

$$r_{ij}+r_{jk}+r_{ki}=r_{ik}+r_{kj}+r_{ji},\quad \forall i,j,k=1,2,\ldots,n. \tag{2.3}$$

Liu et al. (2012) established that $R=(r_{ij})_{n\times n}$ is additively consistent if and only if there exists a normalized priority weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$, $\omega_i\geq 0,\ i=1,2,\ldots,n$ and $\sum_{i=1}^{n}\omega_i=1$, such that

$$r_{ij}=c(\omega_i-\omega_j)+0.5,\quad \forall i,j=1,2,\ldots,n \tag{2.4}$$

where $c=\max\left\{0.5,\ \frac{n}{2}-\min_{1\leq i\leq n}\left\{\sum_{k=1}^{n}r_{ik}\right\}\right\}$.

As $r_{ii}=0.5$ for all $i=1,2,\ldots,n$, one has

$$0.5\leq c\leq \frac{n-1}{2} \tag{2.5}$$

It should be noted that multiple normalized priority weight vectors and c values under (2.4) may exist for a given additively consistent fuzzy preference relation. Conversely, for a given priority weight vector, (2.4) may lead to different additively consistent fuzzy preference relations by setting different c values.

Tanino (1984)'s transformation relation between R and ω is established by setting $c=0.5$ in (2.4). It has been found that this relation does not always hold true (Liu et al., 2012). Fedrizzi and Brunelli (2009), for instance, indicated that the priority weights should not be normalized. To derive normalized priority weights from fuzzy preference relations, the c value in (2.4) is assumed to be $n/2$ by Xu et al. (2009), and revised to be $(n-1)/2$ by Xu et al. (2010; 2014) and Hu et al. (2014).

Elements in a fuzzy preference relation are given from the viewpoint of membership degrees without considering a DM's hesitancy in judgment. To express the DM's hesitancy, Xu (2007) introduced the concept of IPRs.

An IPR on X is denoted by a pairwise intuitionistic judgment matrix

$\tilde{R}=(\tilde{r}_{ij})_{n\times n}=\left((\mu_{ij},v_{ij})\right)_{n\times n}$, where (μ_{ij},v_{ij}) is an intuitionistic fuzzy preference of alternative x_i over x_j such that

$$0\le \mu_{ij},v_{ij}\le 1, 0\le \mu_{ij}+v_{ij}\le 1, \mu_{ij}=v_{ji}, v_{ij}=\mu_{ji}, \mu_{ii}=v_{ii}=0.5 \quad \forall i,j=1,2,...,n \tag{2.6}$$

Each element $\tilde{r}_{ij}=(\mu_{ij},v_{ij})$ in $\tilde{R}$ is an intuitionistic fuzzy number, indicating x_i is preferred to x_j with a degree of μ_{ij}, x_i is non-preferred to x_j with a degree of v_{ij}, and the DM's hesitation in preference between x_i and x_j is determined as $1-\mu_{ij}-v_{ij}$. Especially, $(\mu_{ij},v_{ij})=(0,0)$ indicates a completely unknown preference between x_i and x_j.

By using intuitionistic fuzzy judgments in $\tilde{R}$, Wang (2013) introduced the notion of additively consistent IPRs.

Definition 2.2 (Wang, 2013) An IPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij}=(\mu_{ij},v_{ij})$ is additively consistent if it satisfies the following additive transitivity:

$$\mu_{ij}+\mu_{jk}+\mu_{ki}=\mu_{kj}+\mu_{ji}+\mu_{ik}, \quad \forall i,j,k=1,2,...,n \tag{2.7}$$

By the intuitionistic reciprocal property $\mu_{ij}=v_{ji}, v_{ij}=\mu_{ji}$, $i,j=1,2,...,n$, one has

$$v_{ij}+v_{jk}+v_{ki}=v_{kj}+v_{ji}+v_{ik}, \quad \forall i,j,k=1,2,...,n \tag{2.8}$$

Obviously, if all intuitionistic judgments $\tilde{r}_{ij}=(\mu_{ij},v_{ij})$ are degraded to fuzzy judgments, i.e., $\mu_{ij}+v_{ij}=1$ for all i, j = 1, 2, …, n, the IPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ is reduced to a fuzzy preference relation $R=(r_{ij})_{n\times n}$ with $r_{ij}=\mu_{ij}=1-v_{ij}$. In this case, the additive transitivity (2.7) and (2.8) are reduced to (2.3), equivalent to Tanino's additive transitivity (2.2) for fuzzy preference relations.

3. Estimating missing values in an incomplete intuitionistic preference relation

This section defines an incomplete IPR and its additive consistency and, then develops a two-stage quadratic program method for estimating missing values in incomplete IPRs.

3.1 Additive consistency of incomplete intuitionistic preference relations

A complete IPR $\tilde{R}$ consists of $n(n-1)/2$ intuitionistic judgments over the alternative set X, distributed in the upper (or lower) triangular of $\tilde{R}$. However, in many practical decision situations, a DM is unable or unwilling to provide all of these $n(n-1)/2$ elements, especially when a large number of alternatives have to be evaluated. In this case, the IPR $\tilde{R}$ provided by the DM is incomplete with unknown or missing values in membership and/or nonmembership degrees of judgments in $\tilde{R}$.

Let

$$\Omega = \{(i, j) \mid i, j = 1, 2, ..., n\} \tag{3.1}$$

$$K_{\tilde{R}}^{\mu} = \{(i, j) \mid \mu_{ij} \text{ in } \tilde{R} \text{ is known}, i, j = 1, 2, ..., n, i \neq j\} \tag{3.2}$$

$$K_{\tilde{R}}^{\mu\nu} = \{(i, j) \mid (\mu_{ij}, \nu_{ij}) \text{ in } \tilde{R} \text{ is known}, i, j = 1, 2, ..., n, i \neq j\} \tag{3.3}$$

Definition 3.1 Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \left((\mu_{ij}, \nu_{ij})\right)_{n\times n}$ be a pairwise comparison matrix, if $\Omega - K_{\tilde{R}}^{\mu} - \{(i, j) \mid i = j = 1, 2, ..., n\}$ is a nonempty set and $\tilde{R}$ satisfies

$$\mu_{ii} = \nu_{ii} = 0.5, \forall i = 1, 2, ..., n, \tag{3.4}$$

$$0 \leq \mu_{ij}, \nu_{ij} \leq 1, 0 \leq \mu_{ij} + \nu_{ij} \leq 1, \mu_{ij} = \nu_{ji}, \nu_{ij} = \mu_{ji}, \forall (i, j) \in K_{\tilde{R}}^{\mu\nu}, \tag{3.5}$$

$$0 \leq \mu_{ij} \leq 1, \nu_{ji} = \mu_{ij}, \forall (i, j) \in K_{\tilde{R}}^{\mu} - K_{\tilde{R}}^{\mu\nu}. \tag{3.6}$$

then $\tilde{R}$ is called an incomplete IPR.

Note that Definition 3.1 differs from the existing definitions of incomplete IPRs in Xu (2007), Xu et al. (2011) and Wu and Chiclana (2014), where the membership and nonmembership degrees of a missing element in $\tilde{R}$ must be both unknown.

The additive consistency of IPRs in Definition 2.2 is extended to the incomplete case as follows.

Definition 3.2 Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an incomplete IPR, $\tilde{R}$ is additively consistent, if there exists $\hat{\mu}_{ij}$ for all $i, j = 1, 2, ..., n$ such that

$$\hat{\mu}_{ij} + \hat{\mu}_{jk} + \hat{\mu}_{ki} = \hat{\mu}_{kj} + \hat{\mu}_{ji} + \hat{\mu}_{ik}, \quad i, j, k = 1, 2, ..., n \tag{3.7}$$

$$\hat{\mu}_{ij}^{l} \leq \hat{\mu}_{ij} \leq \hat{\mu}_{ij}^{u}, \quad i, j = 1, 2, ..., n \tag{3.8}$$

$$\hat{\mu}_{ij} + \hat{\mu}_{ji} \leq 1, \quad i, j = 1, 2, ..., n, i \neq j, (i, j) \notin K_{\tilde{R}}^{\mu}, (j, i) \notin K_{\tilde{R}}^{\mu} \tag{3.9}$$

where $\hat{\mu}_{ij}^{l}$ and $\hat{\mu}_{ij}^{u}$ are, respectively, defined by:

$$\widehat{\mu}_{ij}^{l}=\begin{cases}\mu_{ij} & (i,j)\in K_{\tilde{R}}^{\mu}\\ 0.5 & i=j\\ 0 & (i,j)\notin K_{\tilde{R}}^{\mu}\end{cases},\qquad \widehat{\mu}_{ij}^{u}=\begin{cases}\mu_{ij} & (i,j)\in K_{\tilde{R}}^{\mu}\\ 0.5 & i=j\\ 1-\mu_{ji} & (i,j)\notin K_{\tilde{R}}^{\mu},(j,i)\in K_{\tilde{R}}^{\mu}\\ 1 & (i,j)\notin K_{\tilde{R}}^{\mu},(j,i)\notin K_{\tilde{R}}^{\mu}\end{cases} \tag{3.10}$$

It is obvious that $\widehat{\tilde{R}}=(\widehat{\tilde{r}}_{ij})_{n\times n}=\left((\widehat{\mu}_{ij},\widehat{\mu}_{ji})\right)_{n\times n}$ is an additively consistent and complete IPR if $\tilde{R}$ has additive consistency. Therefore, Definition 3.2 ensures that incomplete IPRs with additive consistency can always be completed and the resulting complete IPRs defined by (3.10) are additively consistent.

3.2 A two-stage quadratic program method for estimating missing values

Definition 3.2 furnishes an approach to obtain a complete IPR with additive consistency from a consistent incomplete IPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}=\left((\mu_{ij},v_{ij})\right)_{n\times n}$, where (3.7) - (3.9) are satisfied. However, if $\tilde{R}$ is inconsistent, (3.7) - (3.9) will not hold. To estimate missing values of inconsistent IPRs, (4.7) has to be relaxed but the inconsistency level of the completed IPR should be minimized. In other words, we shall find $\widehat{\mu}_{ij}$ $(i,j=1,2,...,n)$ to minimize the squared deviation between the two sides of (3.7) under the constraints (3.8) and (3.9).

On the other hand, if $\widehat{\mu}_{ij}+\widehat{\mu}_{ji}\rightarrow 0$, then the intuitionistic fuzzy number $(\widehat{\mu}_{ij},\widehat{\mu}_{ji})$ is too hesitant as $1-\widehat{\mu}_{ij}-\widehat{\mu}_{ji}\rightarrow 1$. It is understandable and widely accepted that highly indeterminate or hesitant judgment information contains no or little value for obtaining a reasonable decision result (Dubois & Prade, 2012). Therefore, it is necessary to control the hesitancy of the estimated judgments within an acceptable threshold.

Based on the aforesaid modeling idea, the following quadratic model is established to estimate missing values for an incomplete IPR $\tilde{R}=(\tilde{r}_{ij})_{n\times n}=\left((\mu_{ij},v_{ij})\right)_{n\times n}$.

$$\begin{aligned}&\min\quad J^{c}=\sum_{i,j,k=1}^{n}\left(\widehat{\mu}_{ij}+\widehat{\mu}_{jk}+\widehat{\mu}_{ki}-\widehat{\mu}_{kj}-\widehat{\mu}_{ji}-\widehat{\mu}_{ik}\right)^{2}\\ &s.t.\begin{cases}\widehat{\mu}_{ij}^{l}\le\widehat{\mu}_{ij}\le\widehat{\mu}_{ij}^{u}, & i,j=1,2,...,n\\ \widehat{\mu}_{ij}+\widehat{\mu}_{ji}\le 1, & (i,j)\notin K_{\tilde{R}}^{\mu},(j,i)\notin K_{\tilde{R}}^{\mu}\\ 1-h\le\widehat{\mu}_{ij}+\widehat{\mu}_{ji}. & (i,j)\notin K_{\tilde{R}}^{\mu}\end{cases}\end{aligned} \tag{3.11}$$

where h $(0\le h\le 1)$ is a hesitancy acceptable threshold, and $\widehat{\mu}_{ij}$ $(i,j=1,2,...,n)$ are decision variables.

In model (3.11), the first two lines of constraints guarantee that $\widehat{\tilde{R}} = (\widehat{\tilde{r}}_{ij})_{n\times n} = \left((\widehat{\mu}_{ij}, \widehat{\mu}_{ji})\right)_{n\times n}$ is an IPR, the last line of inequalities ensures that the hesitation degree of the estimated intuitionistic judgment $(\widehat{\mu}_{ij}, \widehat{\mu}_{ji}), (i,j) \notin K_{\tilde{R}}^{\mu}$ is controlled within the acceptable threshold *h*.

By (3.8) and (3.10), one can obtain $\widehat{\mu}_{ii} = 0.5, \forall i = 1, 2, ..., n$ and $\widehat{\mu}_{ij} = \mu_{ij}, \forall (i,j) \in K_{\tilde{R}}^{\mu}$. On the other hand, $\widehat{\mu}_{ij} + \widehat{\mu}_{jk} + \widehat{\mu}_{ki} - \widehat{\mu}_{kj} - \widehat{\mu}_{ji} - \widehat{\mu}_{ik} = 0$ if all or any two of the indexes i, j, k are equal, and $\widehat{\mu}_{ij} + \widehat{\mu}_{jk} + \widehat{\mu}_{ki} - \widehat{\mu}_{kj} - \widehat{\mu}_{ji} - \widehat{\mu}_{ik}$ is a constant if $(i,j), (j,k), (k,i), (i,k), (k,j), (j,i) \in K_{\tilde{R}}^{\mu}$. Consequently, (3.11) can be transformed to the following equivalent quadratic model.

$$\min \ J^{c} = \sum_{\substack{(i,j)\notin K_{\tilde{R}}^{\mu},\\ i\neq j}} \sum_{\substack{k=1,\\ k\neq i, k\neq j.}}^{n} \left(\mu_{ij} + \mu_{jk} + \mu_{ki} - \mu_{kj} - \mu_{ji} - \mu_{ik}\right)^2$$
$$s.t. \begin{cases} 0 \le \mu_{ij} \le 1, & (i,j) \notin K_{\tilde{R}}^{\mu}, i \neq j \\ \mu_{ij} + \mu_{ji} \le 1, & (i,j) \notin K_{\tilde{R}}^{\mu}, i \neq j \\ 1 - h \le \mu_{ij} + \mu_{ji}. & (i,j) \notin K_{\tilde{R}}^{\mu}, i \neq j \end{cases} \tag{3.12}$$

where μ_{ij} $((i,j) \notin K_{\tilde{R}}^{\mu}, i \neq j)$ are decision variables.

Obviously, for any threshold value *h* ($0 \le h \le 1$), the values $\hat{\mu}_{ij}$ $((i,j) \notin K_{\tilde{R}}^{\mu}, i \neq j)$ satisfy the constraints of the model (3.12), where

$$\hat{\mu}_{ij} = \begin{cases} 1 - \mu_{ji} & (i,j) \notin K_{\tilde{R}}^{\mu}, (j,i) \in K_{\tilde{R}}^{\mu} \\ 0.5 & (i,j) \notin K_{\tilde{R}}^{\mu}, (j,i) \notin K_{\tilde{R}}^{\mu}, i \neq j \end{cases} \tag{3.13}$$

Therefore, at least one optimal solution exists for (3.12) for any acceptable hesitancy threshold *h* ($0 \le h \le 1$).

It is easy to prove that the optimal solution to (3.12) has the following property.

Theorem 3.1 If $\widehat{\mu}_{ij}^{*}$ $((i,j) \notin K_{\tilde{R}}^{\mu}, i \neq j)$ is an optimal solution to (3.12), then $\widehat{\mu}_{ij}^{*} + \beta_{ij}$ $((i,j) \notin K_{\tilde{R}}^{\mu}, i \neq j)$ is also an optimal solution to (3.12), where parameters β_{ij} $((i,j) \notin K_{\tilde{R}}^{\mu}, i \neq j)$ satisfy

$$\beta_{ij} = 0, \ (j,i) \in K_{\tilde{R}}^{\mu} \tag{3.14}$$

$$\beta_{ij} = \beta_{ji}, \ 0.5(1 - h - \widehat{\mu}_{ij}^{*} - \widehat{\mu}_{ji}^{*}) \le \beta_{ij} \le 0.5(1 - \widehat{\mu}_{ij}^{*} - \widehat{\mu}_{ji}^{*}), \ (i,j), (j,i) \notin K_{\tilde{R}}^{\mu} \tag{3.15}$$

Theorem 3.1 reveals that numerous solutions may exist for the optimization problems (3.12) when membership and nonmembership degrees of an intuitionistic judgment in $\tilde{R}$ are both unknown. This situation makes it difficult to determine missing values in $\tilde{R}$. Since the missing values have inherent hesitancy in the decision process, it is logical to expect that the completed intuitionistic judgments should properly reflect such hesitancy. For an intuitionistic judgment (μ_{ij}, μ_{ji}), this hesitancy is captured by its hesitation degree $1-\mu_{ij}-\mu_{ji}$. The smaller the accuracy degree $\mu_{ij}+\mu_{ji}$, the stronger this hesitancy. Since model (3.12) controls the hesitation to be within a threshold *h*, to effectively estimate missing values, the second optimization model incorporates the optimal objective value of (3.12) as a constraint to maintain the inconsistency level in the final completed IPR and minimizes the accuracy degrees $\mu_{ij}+\mu_{ji}$ ($(i,j) \notin K_{\tilde{R}}^{\mu}, i \neq j$).

$$\min \quad J_1^c = \sum_{\substack{(i,j)\notin K_{\tilde{R}}^{\mu},\\ i\neq j}} (\mu_{ij}+\mu_{ji})$$
$$s.t.\begin{cases} 0 \le \mu_{ij} \le 1, & (i,j)\notin K_{\tilde{R}}^{\mu}, i\neq j \\ \mu_{ij}+\mu_{ji} \le 1, & (i,j)\notin K_{\tilde{R}}^{\mu}, i\neq j \\ 1-h \le \mu_{ij}+\mu_{ji}, & (i,j)\notin K_{\tilde{R}}^{\mu}, i\neq j \\ \displaystyle\sum_{\substack{(i,j)\notin K_{\tilde{R}}^{\mu},\\ i\neq j}} \sum_{\substack{k=1,\\ k\neq i, k\neq j.}}^{n} \left(\mu_{ij}+\mu_{jk}+\mu_{ki}-\mu_{kj}-\mu_{ji}-\mu_{ik}\right)^2 = J^{c*}. \end{cases} \tag{3.16}$$

where J^{c*} is the optimal objective value of model (3.12), and μ_{ij} ($(i,j)\notin K_{\tilde{R}}^{\mu}, i\neq j$) are decision variables.

Solving the model (3.16), we obtain the optimal solution μ_{ij}^{*} ($(i,j)\notin K_{\tilde{R}}^{\mu}, i\neq j$). Thus, a completed IPR based on $\tilde{R}$ is determined as $\tilde{R}^c = \left((\mu_{ij}^c, v_{ij}^c)\right)_{n\times n}$, where

$$\mu_{ij}^c = \begin{cases} 0.5 & i=j \\ \mu_{ij} & (i,j)\in K_{\tilde{R}}^{\mu} \\ \mu_{ij}^{*} & (i,j)\notin K_{\tilde{R}}^{\mu} \end{cases}, \qquad v_{ij}^c = \begin{cases} 0.5 & i=j \\ \mu_{ji} & (j,i)\in K_{\tilde{R}}^{\mu} \\ \mu_{ji}^{*} & (j,i)\notin K_{\tilde{R}}^{\mu} \end{cases} \tag{3.17}$$

If the incomplete IPR $\tilde{R}$ is additively consistent and $J^{c*}=0$, then $\tilde{R}^c$ satisfies (2.7), implying $\tilde{R}^c$ is additively consistent. It is observed from (3.12), (3.16) and (3.17) that the inconsistency of the completed IPR $\tilde{R}^c$ is maintained at the minimal level obtained from model (3.12) and the overall hesitancy of the estimated intuitionistic judgments in IPR $\tilde{R}^c$ is maximized without exceeding the specified threshold *h*.

4. Quadratic program models for generating interval fuzzy weights

This section proposes a parameterized equation to transform a normalized interval fuzzy weight vector into a consistent IPR and develops two quadratic program models to obtain interval fuzzy weights from complete IPRs.

Denote a normalized interval fuzzy weight vector by $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T = ([\omega_1^l, \omega_1^u], [\omega_2^l, \omega_2^u], ..., [\omega_n^l, \omega_n^u])^T$ with (Sugihara, 2004)

$$0 \le \omega_i^l \le \omega_i^u \le 1,\ \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j{}^l + \omega_i{}^u \le 1,\ \omega_i{}^l + \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j{}^u \ge 1,\ \forall i = 1, 2, ..., n. \tag{4.1}$$

Let

$$\tilde{r}_{ij}^{\bar{\omega}} = (\mu_{ij}^{\bar{\omega}}, v_{ij}^{\bar{\omega}}) = \begin{cases} (0.5, 0.5) & i = j \\ \left(0.5 + \alpha(\omega_i^l - \omega_j^u), 0.5 - \alpha(\omega_i^u - \omega_j^l)\right) & i \ne j \end{cases} \tag{4.2}$$

where α is a parameter such that

$$0.5 \le \alpha \le \frac{n-1}{2},\ 0 \le 0.5 + \alpha(\omega_i^l - \omega_j^u),\ \alpha(\omega_i^u - \omega_j^l) \le 0.5, \quad i \ne j. \tag{4.3}$$

The first inequality in (4.3) comes from (2.5). As α satisfies (4.3) for all $i, j = 1, 2, ..., n, i \ne j$, one can obtain

$$\begin{aligned} &0 \le 0.5 + \alpha(\omega_i^l - \omega_j^u) \le 0.5 + \alpha(\omega_i^u - \omega_j^l) \le 1, \\ &0 \le 0.5 - \alpha(\omega_i^u - \omega_j^l) = 1 - \left(0.5 + \alpha(\omega_i^u - \omega_j^l)\right) \le 1 - \left(0.5 + \alpha(\omega_i^l - \omega_j^u)\right) \le 1, \quad i \ne j. \\ &\left(0.5 + \alpha(\omega_i^l - \omega_j^u)\right) + \left(0.5 - \alpha(\omega_i^u - \omega_j^l)\right) = 1 + \alpha(\omega_i^l - \omega_i^u + \omega_j^l - \omega_j^u) \le 1 \end{aligned}$$

It follows from (4.2) that $0 \le \mu_{ij}^{\bar{\omega}}, v_{ij}^{\bar{\omega}} \le 1$ and $0 \le \mu_{ij}^{\bar{\omega}} + v_{ij}^{\bar{\omega}} \le 1$. Therefore, $\tilde{r}_{ij}^{\bar{\omega}}$ is an intuitionistic fuzzy number.

Theorem 4.1 Assume that $(\mu_{ij}^{\bar{\omega}}, v_{ij}^{\bar{\omega}})$ are defined by (4.2) for all $i, j = 1, 2, ..., n$, then $\tilde{R}_{\bar{\omega}} = (\tilde{r}_{ij}^{\bar{\omega}})_{n \times n} = \left((\mu_{ij}^{\bar{\omega}}, v_{ij}^{\bar{\omega}})\right)_{n \times n}$ is an additively consistent IPR.

***Proof*.** By (4.2), we have $\mu_{ii}^{\bar{\omega}} = v_{ii}^{\bar{\omega}} = 0.5\ \forall i = 1, 2, ..., n$ and

$$\begin{aligned} \mu_{ij}^{\bar{\omega}} &= 0.5 + \alpha(\omega_i^l - \omega_j^u) = 0.5 - \alpha(\omega_j^u - \omega_i^l) = v_{ji}^{\bar{\omega}}, \\ v_{ij}^{\bar{\omega}} &= 0.5 - \alpha(\omega_i^u - \omega_j^l) = 0.5 + \alpha(\omega_j^l - \omega_i^u) = \mu_{ji}^{\bar{\omega}} \end{aligned}$$

for all $i, j = 1, 2, ..., n, i \ne j$. Therefore, as per (2.6), $\tilde{R}_{\bar{\omega}}$ is an IPR.

On the other hand, from (4.2), it follows that

$$\mu_{ij}^{\bar{\varpi}} + \mu_{jk}^{\bar{\varpi}} + \mu_{ki}^{\bar{\varpi}} = 0.5 + \alpha(\omega_i^l - \omega_j^u) + 0.5 + \alpha(\omega_j^l - \omega_k^u) + 0.5 + \alpha(\omega_k^l - \omega_i^u) =$$
$$0.5 + \alpha(\omega_i^l - \omega_k^u) + 0.5 + \alpha(\omega_k^l - \omega_j^u) + 0.5 + \alpha(\omega_j^l - \omega_i^u) = \mu_{kj}^{\bar{\varpi}} + \mu_{ji}^{\bar{\varpi}} + \mu_{ik}^{\bar{\varpi}}$$

for all $i, j, k = 1, 2, ..., n$. By Definition 2.2, $\tilde{R}_{\bar{\varpi}}$ is additively consistent. ■

Theorem 4.1 demonstrates that many additively consistent IPRs may be obtained from a given normalized interval fuzzy weight vector by setting different parameter values for α. Conversely, the same interval fuzzy weight vector may be generated from different IPRs.

As per Theorem 4.1, the following corollary can be directly obtained.

Corollary 4.1 Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \left((\mu_{ij}, v_{ij})\right)_{n\times n}$ be a complete IPR, if there exists a positive parameter value α satisfying (3.3) and a normalized interval fuzzy weight vector $\bar{\varpi} = (\bar{\varpi}_1, \bar{\varpi}_2, \cdots, \bar{\varpi}_n)^T = ([\omega_1^l, \omega_1^u], [\omega_2^l, \omega_2^u], ..., [\omega_n^l, \omega_n^u])^T$, such that

$$\mu_{ij} = 0.5 + \alpha(\omega_i^l - \omega_j^u), \quad i \neq j \tag{4.4}$$

$$v_{ij} = 0.5 - \alpha(\omega_i^u - \omega_j^l), \quad i \neq j \tag{4.5}$$

then $\tilde{R}$ is additively consistent.

It can be shown that, if all intuitionistic judgments $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})$ and interval fuzzy weights $[\omega_i^l, \omega_i^u]$ are reduced to fuzzy judgments and crisp priority weights, respectively, then (4.4) is degraded to (2.4), the relationship between an additively consistent fuzzy preference relation and its crisp priority weight vector.

Eqs. (4.4) and (4.5) only hold for additively consistent IPRs. If an IPR $\tilde{R}$ furnished by a DM is inconsistent, then the intuitionistic judgments in $\tilde{R}$ cannot be expressed as (4.4) and (4.5). In this case, in order to generate an interval fuzzy priority weight vector from $\tilde{R}$, equations (3.4) and (3.5) are relaxed to allow for some deviations. Obviously, the smaller the squared deviations between the left-hand and right-hand sides, the closer $\tilde{R}$ is to an additively consistent IPR. Therefore, the following quadratic program model is established to generate a normalized interval fuzzy weight vector from IPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \left((\mu_{ij}, v_{ij})\right)_{n\times n}$.

$$\min \quad J=\sum_{i=1}^{n}\sum_{j\neq i, j=1}^{n}\left((\mu_{ij}-0.5-\alpha(\omega_i^l-\omega_j^u))^2+(v_{ij}-0.5+\alpha(\omega_i^u-\omega_j^l))^2\right)$$

$$s.t.\begin{cases} 0\le 0.5+\alpha(\omega_i^l-\omega_j^u),\ \alpha(\omega_i^u-\omega_j^l)\le 0.5, & i\neq j=1,2,\dots,n \\ 0.5\le\alpha\le\dfrac{n-1}{2} \\ \sum\limits_{\substack{j=1\\ j\neq i}}^{n}\omega_j{}^l+\omega_i^u\le 1,\ \omega_i{}^l+\sum\limits_{\substack{j=1\\ j\neq i}}^{n}\omega_j{}^u\ge 1, & i=1,2,\dots,n \\ 0\le\omega_i^l\le\omega_i^u\le 1. & i=1,2,\dots,n \end{cases} \tag{4.6}$$

where the first two lines of inequality constraints correspond to (4.3), the last two lines of constraints guarantee that the derived interval fuzzy weights $[\omega_i^l,\omega_i^u]$ $(i=1,2,\dots,n)$ are normalized, and ω_i^l,ω_i^u $(i=1,2,\dots,n)$ and α are decision variables.

As per intuitionistic reciprocity $\mu_{ji}=v_{ij}$ and $v_{ji}=\mu_{ij}$ for all $i,j=1,2,\dots,n$, we have

$$\mu_{ji}-0.5-\alpha(\omega_j^l-\omega_i^u)=v_{ij}-0.5+\alpha(\omega_i^u-\omega_j^l),\ i\neq j \tag{4.7}$$

$$v_{ji}-0.5+\alpha(\omega_j^u-\omega_i^l)=\mu_{ij}-0.5-\alpha(\omega_i^l-\omega_j^u),\ i\neq j \tag{4.8}$$

$$0\le 0.5+\alpha(\omega_j^l-\omega_i^u)\Leftrightarrow\alpha(\omega_i^u-\omega_j^l)\le 0.5,\ i\neq j \tag{4.9}$$

$$\alpha(\omega_j^u-\omega_i^l)\le 0.5\Leftrightarrow 0\le 0.5+\alpha(\omega_i^l-\omega_j^u),\ i\neq j \tag{4.10}$$

Therefore, solutions to (4.6) are able to be found by solving the following quadratic program model:

$$\min \quad J=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_{ij}-0.5-\alpha(\omega_i^l-\omega_j^u))^2+(v_{ij}-0.5+\alpha(\omega_i^u-\omega_j^l))^2\right)$$

$$s.t.\begin{cases} 0\le 0.5+\alpha(\omega_i^l-\omega_j^u),\ \alpha(\omega_i^u-\omega_j^l)\le 0.5, & i=1,2,\dots,n-1,\ j=i+1,\dots,n \\ 0.5\le\alpha\le\dfrac{n-1}{2}, \\ \sum\limits_{\substack{j=1\\ j\neq i}}^{n}\omega_j{}^l+\omega_i^u\le 1,\ \omega_i{}^l+\sum\limits_{\substack{j=1\\ j\neq i}}^{n}\omega_j{}^u\ge 1, & i=1,2,\dots,n \\ 0\le\omega_i^l\le\omega_i^u\le 1. & i=1,2,\dots,n \end{cases} \tag{4.11}$$

If the optimal objective function value $J^*=0$, then all of the elements in $\tilde{R}$ can be expressed by (4.2). By Corollary 4.1, $\tilde{R}$ is additively consistent. While the modeling idea here is to minimize the deviation between the completed IPR and the constructed IPR with additive consistency, this approach has its limitation: it does not conduct an acceptability test for the completed IPR before generating a priority weight. As pointed out by Li et al. (2015), a highly indeterminate comparison matrix can be unacceptable even if it is perfectly consistent as it may contain little or no useful decision information. It is a worthy topic to extend the acceptability notion in Li et al. (2015) to the case of IPRs.

Multiple solutions may be obtained for model (4.11). To obtain a sensible decision result, we need to find a benchmark of these solutions such that the DM's opinions in $\tilde{R}$ can be sufficiently reflected by its corresponding interval fuzzy weights. From (4.4) and (4.5), it is apparent that the closer the parameter α is to 1, the closer the interval fuzzy weight vector is to a consistent IPR. Therefore, it is reasonable to choose a solution from optimal solutions to model (4.11) such that $(\alpha-1)^2$ is minimized. In so doing, the resulting inconsistency is maintained at the same level. Based on this idea, the following quadratic program model is established to obtain such a benchmark.

$$\min \quad J_1=(\alpha-1)^2$$
$$s.t.\begin{cases} 0.5\le\alpha\le\dfrac{n-1}{2}, & \\ \displaystyle\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_{ij}-0.5-\alpha(\omega_i^l-\omega_j^u))^2+(v_{ij}-0.5+\alpha(\omega_i^u-\omega_j^l))^2\right)=J^*, & \\ 0\le 0.5+\alpha(\omega_i^l-\omega_j^u),\ \alpha(\omega_i^u-\omega_j^l)\le 0.5, & i=1,2,\ldots,n-1,\ j=i+1,\ldots,n \\ \displaystyle\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j{}^l+\omega_i{}^u\le 1,\ \omega_i{}^l+\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j{}^u\ge 1, & i=1,2,\ldots,n \\ 0\le\omega_i^l\le\omega_i^u\le 1. & i=1,2,\ldots,n \end{cases} \tag{4.12}$$

where J^* is the optimal objective function value to model (4.11), and ω_i^l,ω_i^u $(i=1,2,\ldots,n)$ are decision variables.

Solving (4.12) yields the optimal value α^* and the optimal interval fuzzy weight vector $\bar{\omega}^*=(\bar{\omega}_1^*,\bar{\omega}_2^*,\cdots,\bar{\omega}_n^*)^T=\left([\omega_1^{*l},\omega_1^{*u}],[\omega_2^{*l},\omega_2^{*u}],\ldots,[\omega_n^{*l},\omega_n^{*u}]\right)^T$.

Example 1: Consider the following 3×3 complete IPR:

$$\tilde{R}_1=(\tilde{r}_{ij})_{3\times3}=\left((\mu_{ij},v_{ij})\right)_{3\times3}=\begin{bmatrix}(0.5,0.5) & (0,1) & (0,1)\\ (1,0) & (0.5,0.5) & (0.5,0.5)\\ (1,0) & (0.5,0.5) & (0.5,0.5)\end{bmatrix}$$

Clearly, the intuitionistic fuzzy judgments in $\tilde{R}_1$ indicate that x_2 and x_3 are indifferent, and are both absolutely preferred to x_1. By Definition 2.2, one can easily verify that $\tilde{R}_1$ is additively consistent. Since $\mu_{ij}+v_{ij}=1$ for all $i,j=1,2,3$, $\tilde{R}_1$ is equivalent to a fuzzy preference relation $R_1=\left(\mu_{ij}\right)_{3\times 3}$.

By substituting $\tilde{R}_1$ into (4.11) and solving this model, we have $J^*=0$. There exist numerous solutions for this optimization model. One can easily verify that $\left(\omega_1^l,\ \omega_1^u,\ \omega_2^l,\ \omega_2^u,\ \omega_3^l,\ \omega_3^u,\ \alpha\right)^T=\left(d,\ d,\ (1\text{-}d)/2,\ (1\text{-}d)/2,\ (1\text{-}d)/2,\ (1\text{-}d)/2,\ 1/(1\text{-}3d)\right)^T$ ($0\le d<1/3$) are all optimal solutions to (3.11). Obviously, the smaller the value d, the stronger the preference degree of "x_2 and x_3 to x_1". As x_2 and x_3 are equally preferred, and are both absolutely superior to x_1, it is appropriate to determine the priority weights of $\tilde{R}_1$ by setting $d=0$ or $\alpha=1$.

By solving (4.12), one can obtain $\alpha^*=1$ and the optimal interval fuzzy weight vector $\bar{\varpi}^*=\left([0,0],[0.5,0.5],[0.5,0.5]\right)^T$.

If the prioritization methods by Gong et al. (2011), Xu (2012) and Wang (2013) are used to generate priority weights from $\tilde{R}_1$, then we have the results in Table 1.

Table 1. Priority weights generated by different methods based on $\tilde{R}_1$

Model	Reference	Priority weight vector $(\bar{\varpi}_1,\bar{\varpi}_2,\bar{\varpi}_3)^T$
Least square model (28)	Gong et al. (2011)	$([0.0310,0.0310],[0.4845,0.4845],[0.4845,0.4845])^T$
Goal program (30)	Gong et al. (2011)	$([0.0310,0.0310],[0.4845,0.4845],[0.4845,0.4845])^T$
Error analysis Eqs. (13) and (15)	Xu (2012)	$([1/6,\ 1/6],[5/12,\ 5/12],[5/12,\ 5/12])^T$
Goal program (4.7)	Wang (2013)	$([0,\ 0],[0.5,\ 0.5],[0.5,\ 0.5])^T$
(4.11) and (4.12)	This article	$([0,\ 0],[0.5,\ 0.5],[0.5,\ 0.5])^T$

Table 1 shows that these five different methods can generate crisp priority weights and consistent ranking results for this IPR. It is trivial to verify that the additively consistent fuzzy preference relation $R_1=\left(\mu_{ij}\right)_{3\times 3}$ can be expressed as (2.4) by the interval weight vector $\bar{\varpi}^*$

under the condition of $\alpha^* = 1$. However, R_1 cannot be expressed as (2.4) via the priority crisp weights obtained by Gong et al. (2011) or Xu (2012) under the condition of $0.5 \le \alpha \le \frac{n-1}{2} = 1$, implying these weights do not accurately reflect the intensity of preference "x_2 and x_3 being absolutely superior to x_1". In addition, it should be noted that, although Wang (2013)'s approach yields the same priority weight vector as the result derived by the proposed model, its optimal objective function value is greater than 0 for this IPR $\tilde{R}_1$. This is attributed to the fact that the transformation formula in Wang (2013) sets $\alpha = 0.5$. These results demonstrate that the parameterized equation (4.2) properly captures the inherent relationship between additively consistent IPRs and normalized interval fuzzy priority weights, and the proposed models are able to determine the best parameter value for α and derive an appropriate interval fuzzy weight vector from an IPR.

5. An application to group decision making with incomplete intuitionistic preference relations

This section puts forward a procedure to solve GDM problems with incomplete IPRs, and provides an enterprise resource planning (ERP) software product selection problem to illustrate how to apply the proposed models.

Consider a GDM problem with a set of decision alternatives $X = \{x_1, x_2, ..., x_n\}$. Assume that $E = \{e_1, e_2, ..., e_m\}$ is a finite set of experts (i.e., DMs), and the importance weight vector of m experts is $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^T$ with $\sum_{s=1}^{m} \lambda_s = 1$ and $\lambda_s \ge 0$ for all $s = 1, 2, ..., m$. Each expert e_s ($s = 1, 2, ..., m$) employs the pairwise comparison method to provide his/her preferences over the n alternatives as an incomplete IPR $\tilde{R}^{(s)} = (\tilde{r}_{ij}^{(s)})_{n\times n} = \left((\mu_{ij}^{(s)}, v_{ij}^{(s)})\right)_{n\times n}$, where some preference values are unknown and the known values are expressed as intuitionistic fuzzy numbers or membership/ nonmembership degrees. Now, a procedure for solving GDM problems with incomplete IPRs is depicted as follows.

Procedure 1

Step 1. Determine the unknown values of $\tilde{R}^{(s)}$ ($s = 1, 2, ..., m$) by solving the models (3.12) and (3.16), and the complete IPR $\tilde{R}^{(s)c} = (\tilde{r}_{ij}^{(s)c})_{n\times n} = \left((\mu_{ij}^{(s)c}, v_{ij}^{(s)c})\right)_{n\times n}$ is determined as per (3.17) for each $s=1,2,...,m$.

Step 2. Employ the intuitionistic fuzzy weighted averaging operator (Xu & Yager, 2009) to aggregate all individual IPRs $\tilde{R}^{(s)c}$ ($s=1,2,...,m$) as per the DMs' importance weight vector $\lambda=(\lambda_1,\lambda_2,...,\lambda_m)^T$ into a collective IPR $\tilde{R}^G=(\tilde{r}_{ij}^G)_{n\times n}=\left((\mu_{ij}^G,v_{ij}^G)\right)_{n\times n}$, where

$$\mu_{ij}^G=\sum_{s=1}^m \lambda_s \mu_{ij}^{(s)c} \qquad v_{ij}^G=\sum_{s=1}^m \lambda_s v_{ij}^{(s)c} \tag{5.1}$$

Step 3. Solve models (4.11) and (4.12) for an optimal group interval fuzzy weight vector $\bar{\omega}^*=(\bar{\omega}_1^*,\bar{\omega}_2^*,\cdots,\bar{\omega}_n^*)^T=\left([\omega_1^{*l},\omega_1^{*u}],[\omega_2^{*l},\omega_2^{*u}],...,[\omega_n^{*l},\omega_n^{*u}]\right)^T$ for $\tilde{R}^G$.

Step 4. Establish the possibility degree matrix $P=\left(P(\bar{\omega}_i^*\geq\bar{\omega}_j^*)\right)_{n\times n}$ as per the following formula (Xu & Chen, 2008).

$$P(\bar{\omega}_i^*\geq\bar{\omega}_j^*)=\max\left\{1-\max\left(\frac{\omega_j^{+*}-\omega_i^{-*}}{\omega_i^{+*}-\omega_i^{-*}+\omega_j^{+*}-\omega_j^{-*}},0\right),0\right\} \tag{5.2}$$

Step 5. Sum up all values in each row of *P*, one gets

$$\theta_i=\sum_{j=1}^n p_{ij} \quad (i=1,2,...,n). \tag{5.3}$$

Step 6. Rank all decision alternatives as per the decreasing order of θ_i ($i=1,2,...,n$), and "alternative x_i being preferred to x_j" is denoted by $x_i \overset{P(\bar{\omega}_i^*\geq\bar{\omega}_j^*))}{\succeq} x_j$.

Theorem 5.1 Let $\tilde{R}^G=(\tilde{r}_{ij}^G)_{n\times n}=\left((\mu_{ij}^G,v_{ij}^G)\right)_{n\times n}$ be a collective IPR defined by (5.1). If all individual IPRs $\tilde{R}^{(s)c}$ ($s=1,2,...,m$) are additively consistent, then $\tilde{R}^G$ is additively consistent.

*Proof***.** As per Definition 2.2, we have

$\mu_{ij}^{(s)c}+\mu_{jk}^{(s)c}+\mu_{ki}^{(s)c}=\mu_{ik}^{(s)c}+\mu_{kj}^{(s)c}+\mu_{ji}^{(s)c}$, $\forall i,j,k=1,2,...,n, s=1,2,...,m$.

It follows that $\sum_{s=1}^m\lambda_s\mu_{ij}^{(s)c}+\sum_{s=1}^m\lambda_s\mu_{jk}^{(s)c}+\sum_{s=1}^m\lambda_s\mu_{ki}^{(s)c}=\sum_{s=1}^m\lambda_s\mu_{ik}^{(s)c}+\sum_{s=1}^m\lambda_s\mu_{kj}^{(s)c}+\sum_{s=1}^m\lambda_s\mu_{ji}^{(s)c}$ $\forall i,j,k=1,2,...,n$. According to (5.1), one can obtain $\mu_{ij}^G+\mu_{jk}^G+\mu_{ki}^G=\mu_{ik}^G+\mu_{kj}^G+\mu_{ji}^G$ $\forall i,j,k=1,2,...,n$. By Definition 2.2, $\tilde{R}^G$ is additively consistent. ■

Next, the proposed models are applied to a GDM problem concerning evaluation and selection of ERP software products (adapted from Gürbüz et al. (2012)).

Example 2: With increasing market competition and economic globalization, an ERP system is considered as an effective solution for improving productivity of industrial enterprises. As ERP software products differ in customization, pricing, functionality and underlying technology, it is important for the enterprises to carefully evaluate ERP software products before their final selection. Assume that three experts e_s $(s=1,2,3)$ are asked to assess four potential ERP software products x_i $(i=1,2,3,4)$ and their importance weight vector is $\lambda=(\lambda_1,\lambda_2,\lambda_3)^T=(0.25,0.45,0.3)^T$. Each expert e_s $(s=1,2,3)$ conducts pairwise comparison of the four ERP software products and furnishes his/her assessments as the following incomplete IPRs $\tilde{R}^{(s)}=(\tilde{r}_{ij}^{(s)})_{4\times4}=\left((\mu_{ij}^{(s)},\nu_{ij}^{(s)})\right)_{4\times4}$:

$$\tilde{R}^{(1)}=\begin{bmatrix} (0.5,0.5) & (0.35,-) & (0.55,0.25) & - \\ (-,0.35) & (0.5,0.5) & (-,0.55) & (0.1,0.05) \\ (0.25,0.55) & (0.55,-) & (0.5,0.5) & (0.55,0.15) \\ - & (0.05,0.1) & (0.15,0.55) & (0.5,0.5) \end{bmatrix}$$

$$\tilde{R}^{(2)}=\begin{bmatrix} (0.5,0.5) & (0.55,0.1) & (-,0.45) & (0.45,0.35) \\ (0.1,0.55) & (0.5,0.5) & - & (0.2,0.15) \\ (0.45,-) & - & (0.5,0.5) & (0.45,-) \\ (0.35,0.45) & (0.15,0.2) & (-,0.45) & (0.5,0.5) \end{bmatrix}$$

$$\tilde{R}^{(3)}=\begin{bmatrix} (0.5,0.5) & (0.15,0.45) & (0.25,-) & - \\ (0.45,0.15) & (0.5,0.5) & (-,0.15) & - \\ (-,0.25) & (0.15,-) & (0.5,0.5) & (0.55,0.15) \\ - & - & (0.15,0.55) & (0.5,0.5) \end{bmatrix}$$

Taking $\tilde{R}^{(1)}$ as an example, its (1, 2) entry (0.35, –) indicates expert 1's partially missing intuitionistic judgment between ERP product x_1 and x_2, where 0.35 gives his/her assessment of x_1 being preferred to x_2 and "–" means that he/she is unable or unwilling to offer his/her nonmembership judgment. On the other hand, its (1, 3) entry (0.55, 0.25) specifies expert 1's complete intuitionistic assessment between product x_1 and x_3, where 0.55 indicates his/her preference degree of x_1 to x_3 and 0.25 gives his/her non-preference degree of x_1 to x_3. Moreover, the "–" for the (1, 4) element signifies expert 1's inability or unwillingness to offer any preference or non-preference assessment between product x_1 to x_4, resulting in a completely missing element in the judgment matrix. In addition, the differences in the (i, j) entry in the three judgment matrices reveal the three experts' subjective judgments between the *i*th and *j*th product as well as their different levels of knowledge between the two ERP products.

Compared to other methods handling missing intuitionistic judgments such as those put forward by Xu (2007), Xu et al. (2011) and Wu and Chiclana (2014), this proposed framework is able to handle both partially missing and completely missing elements while existing methods cannot deal with IPRs with partially missing elements as they require that an element in an IPR is either completely missing or completely known.

By substituting the incomplete IPRs $\tilde{R}^{(s)}$ $(s=1,2,3)$ into (3.12), the following quadratic programs are established.

$$\begin{aligned} \min \quad J^{(1)} &= 3(\mu_{14}+\mu_{21}-\mu_{41}-0.4)^2+2(\mu_{14}-\mu_{41}-0.7)^2 \\ &\quad +2(\mu_{21}-\mu_{23}+0.5)^2+(\mu_{23}-0.2)^2 \\ s.t. & \begin{cases} 0\le\mu_{14}\le 1,\ 0\le\mu_{21}\le 1,\ 0\le\mu_{23}\le 1,\ 0\le\mu_{41}\le 1, \\ 1-h\le\mu_{14}+\mu_{41}\le 1, 1-h\le\mu_{21}+0.35\le 1, 1-h\le\mu_{23}+0.55\le 1. \end{cases} \end{aligned} \tag{5.4}$$

$$\begin{aligned} \min \quad J^{(2)} &= 3(\mu_{13}+\mu_{32}-\mu_{23}-0.9)^2+2(\mu_{13}-\mu_{43}-0.1)^2 \\ &\quad +2(\mu_{23}-\mu_{43}-\mu_{32}+0.4)^2+(\mu_{32}+\mu_{43}-\mu_{23}+0.05)^2 \\ s.t. & \begin{cases} 0\le\mu_{13}\le 1,\ 0\le\mu_{23}\le 1,\ 0\le\mu_{32}\le 1,\ 0\le\mu_{43}\le 1, \\ 1-h\le\mu_{13}+0.45\le 1, 1-h\le\mu_{23}+\mu_{32}\le 1, 1-h\le\mu_{43}+0.45\le 1. \end{cases} \end{aligned} \tag{5.5}$$

$$\begin{aligned} \min \quad J^{(3)} &= 4(\mu_{14}+\mu_{42}-\mu_{24}-\mu_{41}+0.3)^2+3(\mu_{14}+\mu_{31}-\mu_{41}-0.65)^2 \\ &\quad +2(\mu_{23}+\mu_{31}-0.7)^2+3(\mu_{23}+\mu_{42}-\mu_{24}+0.25)^2 \\ s.t. & \begin{cases} 0\le\mu_{14}\le 1,\ 0\le\mu_{23}\le 1,\ 0\le\mu_{24}\le 1,\ 0\le\mu_{31}\le 1,\ 0\le\mu_{41}\le 1,\ 0\le\mu_{42}\le 1, \\ 1-h\le\mu_{14}+\mu_{41}\le 1, 1-h\le\mu_{23}+0.15\le 1, 1-h\le\mu_{24}+\mu_{42}\le 1, 1-h\le\mu_{31}+0.25\le 1. \end{cases} \end{aligned} \tag{5.6}$$

If the hesitation margins of the estimated judgment values are expected to be no more than 0.6, we can set h = 0.6 in the models (5.4)-(5.6) and (3.16).

Solving (5.4) - (5.6) by the optimization Modelling Software Lingo 11, we obtain their optimal objective function values as follows.

$$J^{(1)*}=0.2286667\ ,\quad J^{(2)*}=0.3942857\ ,\quad J^{(3)*}=0.9945810\times 10^{-28}\ .$$

Solving (3.16) results in the corresponding estimated values:

$$\mu_{14}^{(1)*}=0.4900, \mu_{21}^{(1)*}=0.05, \mu_{23}^{(1)*}=0.4333, \mu_{41}^{(1)*}=0.$$

$$\mu_{13}^{(2)*}=0.3356, \mu_{23}^{(2)*}=0, \mu_{32}^{(2)*}=0.4069, \mu_{43}^{(2)*}=0.$$

$$\mu_{14}^{(3)*}=0.3000, \mu_{23}^{(3)*}=0.2500, \mu_{24}^{(3)*}=0.5001, \mu_{31}^{(3)*}=0.4498, \mu_{41}^{(3)*}=0.1000, \mu_{42}^{(3)*}=0.$$

As per (3.17), the completed IPRs $\tilde{R}^{(k)c} = (\tilde{r}_{ij}^{(k)c})_{4\times4} = \left((\mu_{ij}^{(k)c}, v_{ij}^{(k)c})\right)_{4\times4}$ ($k = 1,2,3$) are obtained as

$$\tilde{R}^{(1)c} = \begin{bmatrix} (0.5, 0.5) & (0.35, 0.05) & (0.55, 0.25) & (0.49, 0) \\ (0.05, 0.35) & (0.5, 0.5) & (0.4333, 0.55) & (0.15, 0.05) \\ (0.25, 0.55) & (0.55, 0.4333) & (0.5, 0.5) & (0.55, 0.15) \\ (0, 0.49) & (0.05, 0.15) & (0.15, 0.55) & (0.5, 0.5) \end{bmatrix}$$

$$\tilde{R}^{(2)c} = \begin{bmatrix} (0.5, 0.5) & (0.55, 0.35) & (0.3356, 0.45) & (0.45, 0.35) \\ (0.35, 0.55) & (0.5, 0.5) & (0, 0.4069) & (0.55, 0.15) \\ (0.45, 0.3356) & (0.4069, 0) & (0.5, 0.5) & (0.45, 0) \\ (0.35, 0.45) & (0.15, 0.55) & (0, 0.45) & (0.5, 0.5) \end{bmatrix}$$

$$\tilde{R}^{(3)c} = \begin{bmatrix} (0.5, 0.5) & (0.15, 0.45) & (0.25, 0.4498) & (0.3, 0.10) \\ (0.45, 0.15) & (0.5, 0.5) & (0.25, 0.15) & (0.5001, 0) \\ (0.4498, 0.25) & (0.15, 0.25) & (0.5, 0.5) & (0.55, 0.15) \\ (0.10, 0.3) & (0, 0.5001) & (0.15, 0.55) & (0.5, 0.5) \end{bmatrix}$$

By (5.1), a collective IPR $\tilde{R}^{G} = (\tilde{r}_{ij}^{G})_{n\times n} = \left((\mu_{ij}^{G}, v_{ij}^{G})\right)_{n\times n}$ is determined as

$$\tilde{R}^{G} = \begin{bmatrix} (0.5, 0.5) & (0.3800, 0.3050) & (0.3635, 0.3999) & (0.4150, 0.1875) \\ (0.3050, 0.3800) & (0.5, 0.5) & (0.1833, 0.3656) & (0.4350, 0.0800) \\ (0.3999, 0.3635) & (0.3656, 0.1833) & (0.5, 0.5) & (0.5050, 0.0825) \\ (0.1875, 0.4150) & (0.0800, 0.4350) & (0.0825, 0.5050) & (0.5, 0.5) \end{bmatrix}$$

Solving the quadratic program (4.11) yields its optimal objective value $J^* = 0.01311484$.

By substituting J^* and $\tilde{R}^{G}$ into (4.12) and solving this model, we obtain the optimal value $\alpha^* = 1$ and the optimal group interval fuzzy weights as:

$\bar{\omega}_1^* = [0.2670, 0.3586], \bar{\omega}_2^* = [0.1707, 0.4131], \bar{\omega}_3^* = [0.2761, 0.4433], \bar{\omega}_4^* = [0.0215, 0.2862]$.

As per (5.2), the possibility degree matrix is determined as

$$P = \begin{bmatrix} 0.5 & 0.5626 & 0.3188 & 0.9461 \\ 0.4374 & 0.5 & 0.3345 & 0.7722 \\ 0.6812 & 0.6655 & 0.5 & 0.9766 \\ 0.0539 & 0.2278 & 0.0234 & 0.5 \end{bmatrix}$$

By (5.3), one can obtain $\theta_1 = 2.3275, \theta_2 = 2.0441, \theta_3 = 2.8233$ and $\theta_4 = 0.8051$. As $\theta_3 > \theta_1 > \theta_2 > \theta_4$, the four ERP software products are ranked as $x_3 \overset{68.12\%}{\succeq} x_1 \overset{56.26\%}{\succeq} x_2 \overset{77.22\%}{\succeq} x_4$.

6. Conclusions

In this paper, we introduce the notion of additive consistency for incomplete IPRs and devise a two-stage quadratic program based framework for estimating missing values for an incomplete IPR. The completed IPR minimizes inconsistency and reflects inherent hesitancy of the missing values by controlling it within an acceptable threshold. A parameterized formula is proposed to transform normalized interval fuzzy weights into IPRs with additive consistency. Two quadratic programs are developed to obtain a normalized interval fuzzy weight vector from a complete IPR. By applying the proposed prioritization and completion models, a procedure is presented for solving GDM problems with incomplete IPRs. A numerical example and a group selection problem are provided to illustrate the proposed models.

IPRs furnished by DMs are assumed to be acceptable from the viewpoints of hesitancy and additive consistency. In real-world decisions, the given intuitionistic judgments may be highly hesitant or inconsistent. Future research is needed to address acceptability and consensus models based on IPRs.

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